

$Sol^3 \times \mathbb{E}^1$ -MANIFOLDS

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ABSTRACT. We outline a classification of  $Sol^3 \times \mathbb{E}^1$ -manifolds with  $\beta_1 = 0$ . (The other  $Sol^3 \times \mathbb{E}^1$ -manifolds are mapping tori, and may be classified in terms of automorphisms of 3-manifold groups.)

A closed 4-manifold  $M$  is homeomorphic to an infrasolvmanifold if and only if  $\chi(M) = 0$  and  $\pi_1(M)$  is torsion free and virtually poly- $Z$ , of Hirsch length 4. Every such group is realised in this way, and  $M$  is determined up to homeomorphism by  $\pi$ . Such manifolds are either mapping tori of self-homeomorphisms of 3-dimensional infrasolvmanifolds or are unions of two twisted  $I$ -bundles over such 3-manifolds. (See Chapter 8 of [4].)

There are six families of 4-dimensional infrasolvmanifolds, corresponding to the geometries  $\mathbb{E}^4$ ,  $Nil^3 \times \mathbb{E}^1$ ,  $Nil^4$ ,  $Sol_{m,n}^4$ ,  $Sol_0^4$  and  $Sol_1^4$  of solvable Lie type. The 74 flat 4-manifolds can be listed, while  $Nil^3 \times \mathbb{E}^1$ - and  $Nil^4$ -manifolds (infrasilmanifolds of dimension 4) were classified in [2]. Every torsion-free, virtually poly- $Z$  group of Hirsch length 4 which is not virtually nilpotent is the fundamental group of a 4-manifold with one of the remaining geometries [5]. Manifolds with geometry  $Sol_{m,n}^4$  (with  $m \neq n$ ) or  $Sol_0^4$  are mapping tori of self-homeomorphisms of the 3-torus  $\mathbb{R}^3/\mathbb{Z}^3$ , and so may be classified in terms of conjugacy classes of matrices in  $GL(3, \mathbb{Z})$ . Partial classification of  $Sol^3 \times \mathbb{E}^1$ - and  $Sol_1^4$ -manifolds were given in [1]. A complete classification of  $Sol_1^4$ -manifolds has recently appeared [7].

We show that  $Sol^3 \times \mathbb{E}^1$ -manifolds have canonical Seifert fibrations over 2-dimensional flat orbifolds. The Seifert structure may be used as a basis for classification, the key elements being the base orbifold  $B$ , the action of  $\beta = \pi_1^{orb}(B)$  on  $N = \pi_1(F)$ , where  $F$  is the general fibre, and an ‘‘Euler class’’  $e(\pi) \in H^2(\beta; N)$ . The manifolds which are mapping tori may also be classified in terms of conjugacy classes of automorphisms. We shall consider the interaction of the Seifert fibrations, mapping torus structure and orientability for such manifolds, but shall not otherwise classify them explicitly. The others all have base orbifold the ‘‘pillowcase’’  $S(2, 2, 2, 2)$ , and we shall show that they

may be classified in terms of matrices. Finally, it follows easily from the classification that all  $Sol^3 \times \mathbb{E}^1$ -manifolds bound.

If  $G$  is a group let  $G'$ ,  $\zeta G$  and  $\sqrt{G}$  be the commutator subgroup, centre and maximal nilpotent normal subgroup, respectively, and let  $I(G)$  be the preimage of the torsion of  $G/G'$  in  $G$ . Let  $X^2(G)$  be the subgroup generated by squares. If  $x \in G$  let  $c_x$  be the automorphism induced by conjugation by  $x$ .

Let  $D_\infty = Z/2Z * Z/2Z$  be the infinite dihedral group, with presentation  $\langle u, v \mid u^2 = v^2 = 1 \rangle$ . Let  $G_2 = \mathbb{Z}^2 \rtimes_{-I} \mathbb{Z}$  be the fundamental group of the ‘‘half-turn’’ flat 3-manifold, with presentation  $\langle u, s, t \mid usu^{-1} = s^{-1}, utu^{-1} = t^{-1} \rangle$ . Then  $\sqrt{G_2} = \langle u^2, s, t \rangle \cong \mathbb{Z}^3$ . The only rank 2 direct summand of  $\sqrt{G_2}$  which is normal in  $G_2$  is  $I(G_2) = \langle s, t \rangle$ .

## 1. SEIFERT FIBRATIONS

The group  $Sol^3 \times \mathbb{R} \cong \mathbb{R}^3 \rtimes \mathbb{R}$  is the identity component of the isometry group  $Isom(Sol^3 \times \mathbb{E}^1)$ . Its nilradical is  $\mathbb{R}^3$ , and its commutator subgroup is the vector group  $\mathbb{R}^2$ . If  $\pi$  is a  $Sol^3 \times \mathbb{E}^1$ -lattice it meets the nilradical in  $\sqrt{\pi} \cong \mathbb{Z}^3$ . It also meets the commutator subgroup in a lattice subgroup  $\cong \mathbb{Z}^2$ .

**Theorem 1.** *Let  $M$  be a  $Sol^3 \times \mathbb{E}^1$ -manifold. Then  $M$  has an essentially unique Seifert fibration, with general fibre  $T$  and base  $B = T, Kb, \mathbb{A}, \mathbb{M}b$  or  $S(2, 2, 2, 2)$ .*

*Proof.* The foliation of  $Sol^3 \times \mathbb{R}$  by planes induces a canonical Seifert fibration  $p : M \rightarrow B$ , with general fibre a torus  $T$  and base  $B$  a flat 2-orbifold. Hence  $\pi$  has a normal subgroup  $N \cong \mathbb{Z}^2$  such that  $\pi/N \cong \beta = \pi_1^{orb}(B)$  is a flat 2-orbifold group. If  $\widehat{M}$  is the finite covering space induced from a torus  $\widehat{B}$  covering  $B$  then  $\beta_1(\widehat{M}) = 2$ . The image of the action in  $Aut(N) \cong GL(2, \mathbb{Z})$  is infinite, and contains a matrix with trace  $> 2$ , since  $\pi$  is not virtually nilpotent. Since  $GL(2, \mathbb{Z})$  is virtually free the base orbifold  $B$  must itself fibre over  $S^1$  or the reflector interval  $\mathbb{I}$ .

Suppose that  $q : M \rightarrow \overline{B}$  is another Seifert fibration and  $\overline{N}$  is the fundamental group of the general fibre. The base  $\overline{B}$  is a flat 2-orbifold, since  $\pi$  is solvable, and again must itself fibre over  $S^1$  or  $\mathbb{I}$ . After passing to a covering of index dividing 4, if necessary, we may assume that  $B$  and  $\overline{B}$  are tori. Since  $\beta_1(M) \leq 2$  for any  $Sol^3 \times \mathbb{E}^1$ -manifold, the images of  $N$  and  $\overline{N}$  in  $\pi/\pi'$  must be finite. Hence  $N$  and  $\overline{N}$  must be commensurable. Since  $\pi_1(T)$  is torsion free we must have  $\overline{N} = \overline{N} \cap N = N$ . Thus the fibration is unique (up to automorphisms of the base).

There are examples with  $B = T, Kb, A, Mb$  or  $S(2, 2, 2, 2)$ . These are the only possibilities. For if  $B = P(2, 2)$  then  $\beta_1(\pi) = 0$ , so  $\pi$  is non-orientable, and  $\pi \cong G *_\nu H$ , where  $G = \langle \nu, u \rangle$ ,  $H = \langle \nu, v \rangle$  and  $\pi/\nu$  has presentation  $\langle u, v | v^2 = (vu^2)^2 = 1 \rangle$ . Hence  $\nu = \langle \sqrt{\pi}, u^2 \rangle$ , and so is orientable. A Mayer-Vietoris argument gives  $\nu \cong \mathbb{Z}^3$ , since  $\beta_1(\pi) = 0$ , and one of  $G$  or  $H$  is orientable, since  $\pi$  is non-orientable. But then  $u$  or  $v$  acts as  $\pm I_2$  on  $N$ , and so  $\pi$  is virtually nilpotent. Since  $P(2, 2)$  covers each of  $\mathbb{D}(2, 2)$ ,  $\mathbb{D}(2, \bar{2}, \bar{2})$  and  $\mathbb{D}(\bar{2}, \bar{2}, \bar{2}, \bar{2})$ , these are also ruled out.  $\square$

Closed  $\text{Nil}^3 \times \mathbb{E}^1$ - and  $\text{Nil}^4$ -manifolds have canonical Seifert fibrations. For these, the images of the fundamental group of the general fibre in  $\pi$  are  $\zeta\sqrt{\pi}$  and  $\zeta_2\sqrt{\pi}$  (the second stage of the upper central series), respectively. In general,  $\text{Nil}^3 \times \mathbb{E}^1$ -manifolds may have many Seifert fibrations, but in the  $\text{Nil}^4$  case the fibration is unique.

The *monodromy* of the fibration is the homomorphism  $\theta : \pi_1^{\text{orb}}(B) \rightarrow \text{Aut}(N) = GL(2, \mathbb{Z})$  induced by conjugation in  $\pi$ . The image  $\theta(\beta)$  is infinite, and contains a matrix with trace  $> 2$ , since  $\pi$  is not virtually nilpotent. Hence it has two ends, since  $\phi$  is solvable and  $GL(2, \mathbb{Z})$  is virtually free.

**Lemma 2.** *Let  $F$  be a finite subgroup of  $G = GL(2, \mathbb{Z})$ . If  $N_G(F)$  is infinite then  $F \leq \{\pm I\}$ .*

*Proof.* If  $P \in F \setminus \{\pm I\}$  then it is conjugate to one of  $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ ,  $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ ,  $\begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix}$ ,  $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$  or  $\begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix}$ . (These have orders 2, 2, 3, 4 and 6, respectively.) In each case  $C_G(P)$  is finite, and so  $C_G(F)$  is finite. Since  $\text{Aut}(F)$  is finite the lemma follows.  $\square$

Since we may assume without loss of generality that  $-I \in F$ , this lemma also follows from the fact that  $PSL(2, \mathbb{Z}) \cong Z/2Z * Z/3Z$ .

**Lemma 3.** *Let  $H < GL(2, \mathbb{Z})$  have two ends. Then either*

- (1)  $H \cong \mathbb{Z}$ ; or
- (2)  $H \cong \mathbb{Z} \oplus \langle -I \rangle$ ; or
- (3)  $H = \langle A, B \rangle$  where  $A^2 = B^2 = I$ ; or
- (4)  $H = \langle A, B, -I \rangle$  where  $A^2 = B^2 = I$ ; or
- (5)  $H = \langle A, B \rangle$  where  $A$  has order 4 and  $B^2 = I$ ; or
- (6)  $H = \langle A, B \rangle$  where  $A$  and  $B$  each have order 4.

*In each case neither  $A$  nor  $B$  is  $-I$ .*

*Proof.* Let  $F$  be the maximal finite normal subgroup of  $H$ . Then  $F \leq \{\pm I\}$ , by Lemma 1. If  $H/F \cong \mathbb{Z}$  then (1) or (2) holds.

Suppose that  $H/F \cong D_\infty$ , and let  $A, B \in H$  represent generators of the free factors of  $D_\infty$ . Then  $A$  and  $B$  each have order dividing 4.

Since  $AB$  has infinite order, neither  $A$  nor  $B$  is  $-I$ . If  $A$  has order 2 then  $\det(A) = -1$ , while if  $A$  has order 4 then  $\det(A) = +1$  and  $A^2 = -I$ , and similarly for  $B$ . Thus if  $F = 1$  then (3) holds, while if  $F = \{\pm I\}$  then (4), (5) or (6) holds.  $\square$

Note that if  $A$  has order 4 then  $A^2 = -I$ .

## 2. SOME CHARACTERISTIC SUBGROUPS

The translation subgroup of  $\pi$  is  $\pi_o = \pi \cap (\text{Sol}^3 \times \mathbb{R})$ , and consists of all  $g \in \pi$  such that  $c_g|_{\sqrt{\pi}}$  has positive eigenvalues [3]. Thus it is a characteristic subgroup of  $\pi$ , and contains both  $\sqrt{\pi}$  and  $X^2(\pi)$ . Since  $\beta_1(\pi/N; \mathbb{F}_2) \leq 3$ , the quotient  $\pi/\pi_o$  has exponent 2 and order dividing 8. (Note that  $\pi_o(\text{Isom}(\text{Sol}^3 \times \mathbb{E}^1)) \cong D_8 \times Z/2Z$  has order 16.) It is easy to see that  $\pi_o \leq \pi^+ = \text{Ker}(w_1(\pi))$ , and that  $[\pi^+ : \pi_o] \leq 4$ .

**Lemma 4.** *Let  $\pi$  be a  $\text{Sol}^3 \times \mathbb{E}^1$ -group. Then  $\zeta\pi_o \cong \mathbb{Z}$ .*

*Proof.* The radical of  $G = \text{Sol}^3 \times R$  is  $\sqrt{G} = R^3$ , and  $G/\sqrt{G} \cong R$  acts on  $\sqrt{G}$  via  $E : R \rightarrow SL(3, R)$ , where  $E(x)$  is the diagonal matrix  $[e^x, 1, e^{-x}]$ . Since  $\pi_o$  is a lattice in  $G$ , the intersection  $G \cap \pi_o$  is a lattice subgroup in  $R^3$ , and  $\pi_o/G \cap \pi_o$  is a discrete subgroup of  $R$ . Moreover,  $G \cap \pi_o = \sqrt{\pi}$ . (See §§4.29–31 of [8].) If  $t \in \pi_o$  represents a generator of this quotient then  $E(t)$  is conjugate in  $GL(3, R)$  to a matrix in  $SL(3, \mathbb{Z})$ . Since  $E(t)$  has eigenvalues  $\alpha < 1 < \alpha^{-1}$ , it fixes an infinite cyclic subgroup of  $\sqrt{\pi}$ . This subgroup is clearly  $\zeta\pi_o$ .  $\square$

The normal subgroup  $N$  of Theorem 1 may be defined intrinsically as  $I(\pi_o)$ , and so is also a characteristic subgroup of  $\pi$ .

**Theorem 5.** *Let  $\pi$  be a  $\text{Sol}^3 \times \mathbb{E}^1$ -group. Then*

- (1) *there is an unique maximal, infinite cyclic, normal subgroup  $C$  containing  $\zeta\pi_o$ ;*
- (2)  *$N$  is the unique maximal, rank 2 abelian, normal subgroup;*
- (3)  *$N, C < \sqrt{\pi}$ ,  $N \cap C = 1$  and  $N \oplus C$  has finite index in  $\sqrt{\pi}$ ;*
- (4)  *$\zeta\pi \cap \pi' = 1$ ;*
- (5)  *$\zeta\pi \neq 1$  if and only if  $\pi \cong \sigma \rtimes \mathbb{Z}$ , where  $\sigma$  is a  $\text{Sol}^3$ -group.*

*Proof.* If  $C_1$  and  $C_2$  are two infinite cyclic, normal subgroups containing  $\zeta\pi_o$ , then  $C_1C_2$  is virtually infinite cyclic, and therefore infinite cyclic, since  $\pi$  is torsion free. Since  $\pi$  is virtually polycyclic, every increasing chain of subgroups is finite, and so there is an unique maximal such subgroup  $C$ .

The subgroup  $N$  is maximal since  $\pi/N$  has no finite normal subgroups. If  $\tilde{N}$  is a rank 2 abelian normal subgroup such that  $\tilde{N} \cap N = 1$

then  $N\tilde{N}$  is nilpotent of Hirsch length 4. Therefore  $N$  has finite index in  $\tilde{N}N$ , since no infinite cyclic subgroup of  $N$  is normal in  $\pi$ . Since  $\pi/N$  has no finite normal subgroups it follows that  $\tilde{N} \leq N$ , and so  $N$  is the unique maximal, rank 2 abelian, normal subgroup of rank 2.

Since  $N$  and  $C$  are nilpotent normal subgroups,  $N < \sqrt{\pi}$  and  $C < \sqrt{\pi}$ . If  $N \cap C \neq 1$  then  $\pi$  would be virtually abelian. Therefore  $N \cap C = 1$  and so  $N \oplus C \leq \sqrt{\pi}$ . The subgroup  $N$  is a direct summand of  $\sqrt{\pi}$ , since  $\pi/N$  has no finite normal subgroups, and so  $\sqrt{\pi}/(N \oplus C)$  is finite cyclic.

If  $\zeta\pi \neq 1$  then  $\zeta\pi \leq C$ , and so  $\zeta(\pi/N)$  is infinite. Hence the base orbifold must be  $T, Kb, \mathbb{A}$  or  $\mathbb{M}b$ , and so  $\zeta\pi \cap \pi' = 1$ . Let  $p : \pi \rightarrow \mathbb{Z}$  be an epimorphism which maps  $\zeta\pi$  nontrivially, and let  $\kappa = \text{Ker}(p)$ . Then  $\pi \cong \kappa \rtimes \mathbb{Z}$ , and is virtually  $\kappa \times \mathbb{Z}$ . Hence  $\kappa$  is a  $\text{Sol}^3$  group. Conversely, if  $\pi \cong \sigma \rtimes \mathbb{Z}$ , where  $\sigma$  is a  $\text{Sol}^3$ -group, then  $\zeta\pi \cong \mathbb{Z}$ .  $\square$

The subgroup  $C$  may not be maximal in  $\sqrt{\pi}$ , and  $\zeta\pi$  need not be a direct factor of  $\pi$ , although  $\text{Sol}^3 \times R$  is a direct product. For instance, if  $\Psi = \begin{pmatrix} a & b \\ c & a \end{pmatrix}$ , where  $a > 1$  and  $a^2 - bc = 1$  then  $|\det(I - \Psi)| = 2(a - 1) > 1$ . If the column vector  $\xi \in \mathbb{Z}^2$  is not in  $\text{Im}(I - \Psi)$  then the bordered matrix  $\Theta = \begin{pmatrix} 1 & 0 \\ \xi & \Psi \end{pmatrix} \in SL(3, \mathbb{Z})$  is not conjugate to a block diagonal matrix, and  $\mathbb{Z}^3 \rtimes_{\Theta} \mathbb{Z}$  is a discrete cocompact subgroup of  $\text{Sol}^3 \times R$ , but is not a direct product.

For another example, let  $\beta = \pi_1(Kb) = \langle x, y \mid xyx^{-1} = y^{-1} \rangle$ , and define  $\theta : \beta \rightarrow GL(2, \mathbb{Z})$  by  $\theta(x) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$  and  $\theta(y) = \Psi$  (as above). Then  $\pi = \mathbb{Z}^2 \rtimes_{\theta} \beta$  is a  $\text{Sol}^3 \times \mathbb{E}^1$ -group, with centre  $\langle x^2 \rangle$ .

### 3. MAPPING TORI

Let  $M$  be a  $\text{Sol}^3 \times \mathbb{E}^1$ -manifold, and let  $\pi = \pi_1(M)$ . Partial classifications of certain classes of such groups have been given in [1]. If  $\pi/\pi'$  is infinite then  $\pi \cong \nu \rtimes \mathbb{Z}$ , where either  $\sqrt{\pi} \leq \nu$  and  $[\nu : \sqrt{\pi}] \leq 2$  (by Theorem 8.4 of [4]) or  $\nu$  is the group of a  $\text{Sol}^3$ -manifold. All such semidirect products may be classified in terms of conjugacy classes in  $\text{Out}(\nu)$ . In this section we shall consider the interactions between Seifert fibrations, mapping tori and orientability for these groups. We shall consider the groups with  $\pi/\pi'$  finite in the next section.

In Theorem 1 it was shown that the image of  $N$  in  $\pi/\pi'$  is finite. Hence  $\beta_1(\pi) \leq 2$ , and  $\beta_1(\pi) = 2$  if and only if  $B = T$ . If so, the image of the action in  $GL(2, \mathbb{Z})$  is  $\mathbb{Z}$  or  $\mathbb{Z} \oplus \langle -I \rangle$ , by Lemma 3, and so  $\pi \cong \nu \rtimes \mathbb{Z}$ , where  $\nu = \sqrt{\pi}$  or  $G_2$ . The group  $\pi$  is also a semidirect product  $\sigma \rtimes \mathbb{Z}$ , where  $\sigma$  is a  $\text{Sol}^3$ -group (with  $\sigma/\sqrt{\sigma} \cong \mathbb{Z}$ ), in infinitely many ways. (However,  $\pi$  need not be a direct product  $\sigma \times \mathbb{Z}$ .) There are orientable examples and non-orientable examples. (All  $T$ -bundles

over  $T$  have been classified, in terms of extension data [9]. However Proposition 3 of [9] appears to overlook some cases.)

If  $\beta(\pi) = 1$  then  $B = Kb, \mathbb{A}$  or  $\mathbb{M}b$ , and the splitting is unique. If  $B = Kb$  there are orientable and non-orientable examples with  $\pi \cong \sqrt{\pi} \rtimes \mathbb{Z}$  and with  $\pi \cong \sigma \rtimes \mathbb{Z}$ , where  $\sigma$  is a  $\text{Sol}^3$ -group. (See §8 of Chapter 8 of [4].)

**Lemma 6.** *If  $\nu = \mathbb{Z}^3$  or  $G_2$  and  $\pi \cong \nu \rtimes \mathbb{Z}$  is the group of a  $\text{Sol}^3 \times \mathbb{E}^1$ -manifold  $M$  then  $M$  is Seifert fibred over  $T$  or  $Kb$ .*

*Proof.* In either case  $N < \sqrt{\pi} \leq \nu$ , and  $\pi/N$  has no nontrivial finite normal subgroup. If  $\nu = \mathbb{Z}^3$  then  $\nu/N$  is abelian of rank 1. If  $\nu = G_2$  then  $N = I(\nu)$ , since  $I(G_2)$  is characteristic in  $G_2$ , and hence in  $\pi$ . and so  $\nu/N \cong \mathbb{Z}$ .  $\square$

**Lemma 7.** *Let  $\sigma$  be a  $\text{Sol}^3$ -group such that  $\sigma/\sqrt{\sigma} \cong D_\infty$ . Then  $\sigma$  is orientable, and automorphisms of  $\sigma$  are orientation preserving.*

*Proof.* The hypotheses imply that  $\sigma/\sigma'$  is finite. Thus  $H_1(\sigma; \mathbb{Q}) = 0$ . Since  $\sigma$  is a  $PD_3$ -group,  $\chi(\sigma) = 0$ , Therefore  $H_3(\sigma; \mathbb{Q}) \neq 0$ , and so  $\sigma$  is orientable. (This can also be deduced from the fact that if  $N \in GL(2, \mathbb{C})$  is conjugate to  $N^{-1}$  then either  $\det(N) = 1$  or  $N^2 = 1$ .) Let  $[\sigma] \in H_3(\sigma; \mathbb{Z})$  be a generator.

Let  $u$  and  $v \in \sigma$  represent generating involutions of  $D_\infty$ , and let  $t = uv$ . Let  $f$  be an automorphism of  $\sigma$ . Then  $f$  restricts to an automorphism of  $\sqrt{\sigma}$ , and induces an automorphism of  $\sigma/\sqrt{\sigma}$ . After composition with an inner automorphism of  $\sigma$ , if necessary, we may assume that either  $f(u) \equiv u$  and  $f(v) \equiv v$ , or  $f(u) \equiv v$  and  $f(v) \equiv u \pmod{\sqrt{\sigma}}$ . Let  $P = f|_{\sqrt{\sigma}}$ , and suppose that  $f(t) \equiv t^\epsilon \pmod{\sqrt{\sigma}}$ . Then  $f_*[\sigma] = \epsilon \det(P)[\sigma]$ .

In the first case,  $f(t) \equiv t \pmod{\sqrt{\sigma}}$ , while  $Pc_u|_{\sqrt{\sigma}} = c_u|_{\sqrt{\sigma}}P$  and  $Pc_v|_{\sqrt{\sigma}} = c_v|_{\sqrt{\sigma}}P$ , and so  $P = I$ . In the second case,  $f(t) \equiv t^{-1} \pmod{\sqrt{\sigma}}$ , while  $Pc_u|_{\sqrt{\sigma}}P^{-1} = c_v|_{\sqrt{\sigma}}$  and  $Pc_v|_{\sqrt{\sigma}}P^{-1} = c_u|_{\sqrt{\sigma}}$ . Hence  $P^2 = I$ . Since  $c_t = c_u c_v$  and  $c_t|_{\sqrt{\sigma}}$  has infinite order,  $P \neq \pm I$ . Therefore  $\det P = -1$ . In each case,  $f$  is orientation preserving.  $\square$

Let  $B_1 = \mathbb{Z} \times Kb$  and  $B_2$  be the non-orientable flat 3-manifold groups with holonomy  $\mathbb{Z}/2\mathbb{Z}$ .

**Theorem 8.** *Let  $M$  be a  $\text{Sol}^3 \times \mathbb{E}^1$ -manifold which is Seifert fibred over  $B = \mathbb{A}$  or  $\mathbb{M}b$ , and let  $\pi = \pi_1(M)$ . Then  $M$  is orientable if and only if  $\pi \cong \sigma \rtimes \mathbb{Z}$ , where  $\sigma$  is a  $\text{Sol}^3$ -group such that  $\sigma/\sqrt{\sigma} \cong D_\infty$ . If  $M$  is non-orientable then  $\pi \cong B_1 \rtimes \mathbb{Z}$ , and  $B = \mathbb{A}$ .*

*Proof.* Since  $\beta_1(M) = 1$  there is a unique splitting  $\pi = \nu \rtimes_\theta \mathbb{Z}$ , and  $\nu$  is an extension of  $D_\infty$  by  $N = \mathbb{Z}^2$ .

If  $\nu$  is a  $\text{Sol}^3$ -group then  $N = \sqrt{\nu}$ , since  $\sqrt{\nu}$  is characteristic and  $\nu/\sqrt{\nu}$  has no nontrivial finite normal subgroup. Since  $\nu$  is orientable and  $\theta$  is orientation preserving, by Lemma 7,  $M$  is orientable.

If  $\nu$  is virtually abelian then it has holonomy  $Z/2Z$ , and is non-orientable, by Lemma 6. Therefore  $\nu = B_1$ , since  $B_2$  does not map onto  $D_\infty$ . Hence  $N = \zeta B_1 \cong \mathbb{Z}^2$ , since  $\zeta D_\infty = 1$  and  $B_1/\zeta B_1 \cong D_\infty$ . Automorphisms of  $B_1$  do not swap the generators of  $D_\infty$ . The corresponding mapping tori are Seifert fibred over  $\mathbb{A}$ , but not over  $\mathbb{Mb}$ . Since  $\nu$  is non-orientable,  $M$  is non-orientable.  $\square$

There are examples of each type allowed by Theorem 8. For instance, let  $\sigma$  be the  $\text{Sol}^3$ -group with presentation

$$\langle x, y, u, v \mid xy = yx, u^2 = x, uyu^{-1} = y, v^2 = x^3y^{-2}, vxv^{-1} = x^{17}y^{-12}, \\ v y v^{-1} = x^{24}y^{-17} \rangle.$$

Then  $\sqrt{\sigma} = \langle x, y \rangle$  and  $\sigma/\sqrt{\sigma} \cong D_\infty$ . We may define an involution  $f$  of  $\sigma$  by  $f(u) = v$ ,  $f(y) = x^4y^{-3}$  and  $f(v) = u$ . The groups  $\sigma \times \mathbb{Z}$  and  $\sigma \rtimes_f \mathbb{Z}$  are groups of orientable  $\text{Sol}^3 \times \mathbb{E}^1$ -manifolds which are Seifert fibred over  $\mathbb{A}$  and  $\mathbb{Mb}$ , respectively.

The group  $B_1$  has a presentation

$$\langle t, x, y \mid tx = xt, ty = yt, xyx^{-1} = y^{-1} \rangle.$$

Let  $\theta(t) = t^3x^2$ ,  $\theta(x) = t^4x^3$  and  $\theta(y) = y$ . Then  $\pi = B_1 \rtimes_\theta \mathbb{Z}$  is the group of a non-orientable  $\text{Sol}^3 \times \mathbb{E}^1$ -manifold which is Seifert fibred over  $\mathbb{A}$ .

#### 4. $\pi/\pi'$ FINITE

Suppose henceforth that  $\beta_1(M) = 0$ . Then  $M$  is Seifert fibred over  $S(2, 2, 2, 2)$ , and  $\pi$  is an extension of  $\beta = \pi_1^{\text{orb}}(S(2, 2, 2, 2)) \cong \mathbb{Z}^2 \rtimes_{-I} Z/2Z$  by  $N = \mathbb{Z}^2$ . If  $\theta$  is the monodromy of the fibration the quotient of  $\theta(\beta)$  by its maximal finite normal subgroup is  $D_\infty$ , since  $\beta/\beta'$  is finite.

**Lemma 9.** *Let  $M$  be a  $\text{Sol}^3 \times \mathbb{E}^1$ -manifold which is Seifert fibred over  $S(2, 2, 2, 2)$ , and let  $\pi = \pi_1(M)$ . Then  $\pi/\sqrt{\pi} \cong D_\infty$ .*

*Proof.* Let  $\theta : \beta \rightarrow GL(2, \mathbb{Z})$  be the monodromy of the Seifert fibration. Since  $\beta/\beta' \cong (Z/2Z)^3$  the image  $\theta(\beta)$  also has finite abelianization, and so is not  $\mathbb{Z}$  or  $\mathbb{Z} \oplus Z/2Z$ . If  $x \in \pi$  has nontrivial image in  $\beta/\beta'$  then the image of  $x$  in  $\beta$  has order 2. Hence  $\theta(\beta)$  must be  $D_\infty$  or  $D_\infty \times Z/2Z$ , by Lemma 3. Since  $\langle N, x \rangle$  is torsion-free, either  $\theta(x) = I$  or  $\theta(x)$  is conjugate in  $GL(2, \mathbb{Z})$  to  $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ . Thus we may exclude type (4) of Lemma 3 also, and so  $\theta(\beta) \cong D_\infty$ .

Let  $u, v \in \pi$  represent  $A, B \in \theta(\beta)$ , and let  $w \in \pi$  represent a generator of  $\text{Ker}(\theta)$ . Then  $\langle N, w \rangle = \sqrt{\pi} \cong \mathbb{Z}^3$ , and so  $\pi$  is an extension of  $D_\infty$  by  $\sqrt{\pi}$ .  $\square$

Thus  $\pi$  gives rise to an extension

$$\xi : 1 \rightarrow \mathbb{Z}^3 \rightarrow \pi \rightarrow D_\infty \rightarrow 1,$$

which is determined by the action  $\theta : D_\infty \rightarrow GL(3, \mathbb{Z})$  and a class  $e(\xi) \in H^2(D_\infty; \mathbb{Z}^3)$ . (Not all classes correspond to infrasolvmanifolds; the semidirect product has torsion elements!) The free product structure of  $D_\infty$  also induces a splitting of  $\pi$  as a free product with amalgamation  $G_A *_{\sqrt{\pi}} G_B$ , where  $G_A = \langle \sqrt{\pi}, u \rangle$  and  $G_B = \langle \sqrt{\pi}, v \rangle$ .

Since  $\sqrt{\pi}$  is characteristic in  $\pi$ , an isomorphism  $f : \pi \rightarrow \tilde{\pi}$  of such groups corresponds to a pair of automorphisms  $f \in GL(3, \mathbb{Z})$  and  $g \in \text{Aut}(D_\infty)$  such that  $f(\theta(d)(z)) = \tilde{\theta}(g(d)(f(z)))$ , for all  $z \in \mathbb{Z}^3$  and  $d = u$  or  $v \in D_\infty$ , and such that  $g^*e(\xi) = f_\#e(\xi)$  in  $H^2(D_\infty; \mathbb{Z}^3)$ .

**Theorem 10.** *Let  $M$  be a  $\text{Sol}^3 \times \mathbb{E}^1$ -manifold which is Seifert fibred over  $S(2, 2, 2, 2)$ . Then  $M$  is determined by a matrix  $\Psi = \begin{pmatrix} a & b \\ -c & a \end{pmatrix}$  in  $SL(2, \mathbb{Z})$  such that  $a$  is odd,  $|a| > 1$ ,  $b$  and  $c$  are even, and  $b > 0$ . A second such matrix  $\Phi$  determines an isomorphic group if and only if  $\Phi = \Psi$  or  $\Psi^{-1}$ .*

*Proof.* Let  $\pi = \pi_1(M)$ . We shall use the notation of Lemma 6 and the above paragraph, and shall normalize the data representing  $\pi$  as an extension of  $D_\infty$  by  $\mathbb{Z}^3$ .

The elements  $u$  and  $v$  each act orientably on  $\sqrt{\pi}$ , since  $uwu^{-1} \equiv v w v^{-1} \equiv w^{-1} \pmod{N}$ , and  $A$  and  $B$  are each conjugate to  $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ . Since  $G_A = \langle \sqrt{\pi}, u \rangle$  and  $G_B = \langle \sqrt{\pi}, v \rangle$  are orientable and have holonomy of order 2, we have  $G_A \cong G_B \cong G_2$ , and so  $\sqrt{G_A} = \sqrt{\pi} = \sqrt{G_B}$ . (Hence  $\pi \cong G_2 *_{\phi} G_2$ , where  $\phi \in \text{Aut}(\sqrt{G_2}) \cong GL(3, \mathbb{Z})$ .) Since  $I(G_A)$  and  $I(G_B)$  are direct summands of  $\sqrt{\pi}$ , so is their intersection  $C = I(G_A) \cap I(G_B)$ , which is the characteristic subgroup of Theorem 5. Therefore  $\sqrt{\pi} = N \oplus C$  splits as a direct sum, in this case.

After a change of basis (compatible with this splitting), if necessary, we may assume that  $v$  acts via  $V = \text{diag}[1, -1, -1]$ . The action of  $u$  is determined by the block diagonal matrix  $U = \begin{pmatrix} A & 0 \\ 0 & -1 \end{pmatrix} \in SL(3, \mathbb{Z})$ , such that  $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$  is conjugate to  $D = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ . Thus  $ad - bc = -1$  and  $a + d = 0$ . Moreover,  $a$  is odd and  $b, c$  are even, since  $D \equiv I \pmod{2}$ . The matrix  $\Psi = DA = \begin{pmatrix} a & b \\ -c & a \end{pmatrix}$  gives the action of  $uv$  on  $N$ . Since  $\pi$  is not virtually nilpotent,  $\Psi$  must have infinite order and distinct eigenvalues, and so  $|\text{tr}(\Psi)| = |a - d| = 2|a| > 2$  and  $bc = 1 - a^2 \neq 0$ .



Since we have diagonalized  $V$ , the matrices  $A$  and  $\Psi$  are well-defined up to conjugation by  $D$ , given the choice of generators for  $D_\infty$ . Conjugation by  $D$  changes only the sign of the off-diagonal elements. This replaces  $\Psi$  by its inverse (as follows also from  $(D\Psi)^2 = A^2 = I$ .) Every pair of generating involutions for  $D_\infty$  is the image of the pair  $(u, v)$  under an automorphism of  $D_\infty$ . Since these either fix or invert  $\Psi$ , the choice of generators does not further enlarge the set of representative matrices. Since  $bc \neq 0$ , there is a unique representative with  $b > 0$ .

Since  $H^2(D_\infty; \mathbb{Z}^3) \cong H^2(Z/2Z; \mathbb{Z}^3) \oplus H^2(Z/2Z; \mathbb{Z}^3) \cong (Z/2Z)^2$ , and the outer automorphisms of  $D_\infty$  acts on this group by the involution that swaps the summands, the extension class is determined by the extension classes corresponding to  $G_A$  and  $G_B$ , which are each the nontrivial class in  $H^3(Z/2Z; \mathbb{Z}^3)$ .  $\square$

**Corollary.** *If  $M$  is a  $\text{Sol}^3 \times \mathbb{E}^1$ -manifold which is Seifert fibred over  $S(2, 2, 2, 2)$  then  $w_1(M)^2 = 0$ .*

*Proof.* The above theorem leads to the presentation

$$\langle u, v, x, y, z \mid uxu^{-1} = x^a y^c, \ uyu^{-1} = x^b y^d, \ uzu^{-1} = z^{-1}, \ u^2 = x^e y^f, \\ v^2 = x, \ vyv^{-1} = y^{-1}, \ vzv^{-1} = z^{-1} \rangle$$

for  $\pi$ , where  $x^e y^f$  generates the  $+1$ -eigenspace of  $U$ . Since  $a, d$  are odd and  $b, c$  are even, the images of  $x, y$  and  $z$  in  $\pi/\pi'$  have order 2, and hence the images of  $u$  and  $v$  in  $\pi/\pi'$  have order 4. Since  $x, y$  and  $z$  are orientation-preserving, while  $u$  and  $v$  are orientation-reversing, it follows that  $w_1(M)$  factors through  $Z/4Z$ , and so  $w_1(M)^2 = 0$ .  $\square$

## 5. $\text{Sol}^3 \times \mathbb{E}^1$ -MANIFOLDS ARE BOUNDARIES

We shall follow the example of [7] by using our classification to show that all  $\text{Sol}^3 \times \mathbb{E}^1$ -manifolds are boundaries. Infrapmanifolds are finitely covered by coset spaces of solvable Lie groups. Such coset spaces are parallelizable, and so the rational Pontrjagin classes of infrapmanifolds are 0. In particular, orientable 4-dimensional infrapmanifolds have signature  $\sigma = 0$ . Therefore they bound orientably, and those with  $w_2 = 0$  bound as *Spin*-manifolds, since  $\Omega_4$  and  $\Omega_4^{\text{Spin}}$  are detected by  $\sigma$ .

Non-orientable bordism is detected by Stiefel-Whitney classes. Here the only Stiefel-Whitney class of interest is  $w_1^4$ , since  $w_4(M) \cap [M]$  is the *mod*-(2) reduction of  $\chi(M) = 0$ , and the other Stiefel-Whitney numbers may be determined by the Wu relations.

**Theorem 11.** *Every  $\text{Sol}^3 \times \mathbb{E}^1$ -manifold  $M$  is a boundary.*

*Proof.* Let  $\pi = \pi_1(M)$  and  $w = w_1(M)$ . If  $M$  fibres over an  $r$ -manifold, with orientable fibre, then  $w^{r+1} = 0$ . Otherwise, either  $\pi \cong B_1 \rtimes \mathbb{Z}$  or  $\beta_1(\pi) = 0$ . If  $\pi \cong B_1 \rtimes \mathbb{Z}$  then  $M$  is the total space of an  $S^1$ -bundle (corresponding to the characteristic subgroup  $B'_1$ ), and so it bounds the associated  $D^2$ -bundle space. Finally, if  $\beta_1(\pi) = 0$  then  $w^2 = 0$ , by the Corollary to Theorem 10.  $\square$

It is well known that all flat manifolds bound. All  $Sol_0^4$ - and  $Sol_{m,n}^4$ -manifolds (with  $m \neq n$ ) are mapping tori of self-diffeomorphisms of the 3-torus, by Corollary 8.5.1 of [4]. Hence they are boundaries. It remains an open question whether  $Nil^3 \times \mathbb{E}^1$ - and  $Nil^4$ -manifolds with  $\beta_1 = 0$  are boundaries.

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